Logic 2: Modal Logic

Lecture 15

Wolfgang Schwarz

University of Edinburgh

Conditionals in natural language

Conditionals in natural language

- If Russia invades Estonia, NATO will attack Russia.
- If we don't reduce greenhouse gases, the climate might get out of control.
- You will be faster if you take a taxi.
- If Heisenberg hadn't undermined the Nazi's nuclear weapons programme, Germany would have won the war.
- If Jones hadn't untied the rope, Smith would not have fallen.

How should we analyze these sentences?

Material conditionals?

Material conditionals?

One possibility: English conditionals are material conditionals.

A	B	if A then B
1	1	1
1	0	0
0	1	1
0	0	1

The logic of material conditionals

		$A \rightarrow B$
Modus Ponens	if *A* then *B*, *A* ∴ *B*	valid
Conditional Proof	*A* entails *B* ∴ if *A* then *B*	valid
Or-to-If	*A* or *B* ∴ if not ¬*A* then *B*	valid
Import-Export	if *A* then if *B* then *C* ∴∴ if *A* and *B* then *C*	valid
Contraposition	if *A* then *B* ∴ if not *B* then not *A*	valid
Transitivity	if *A* then *B*, if *B* then *C* ∴ if *A* then *C*	valid
SDA	if *A* or *B* then *C* ∴ if *A* then *C* and if *B* then *C*	valid
Antec. Strength.	if *A* then *C* ∴ if *A* and *B* then *C*	valid
False Antec.	not *A* ∴ if *A* then *B*	valid
True Cons.	*B* ∴ if *A* then *B*	valid

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Material conditionals?

1. *True Cons.* $B \therefore$ if A then B
   The lecture ends at 2 pm. Therefore: If the building collapses at 1.45 then the lecture ends at 2 pm.
2. *False Antec.* not $A \therefore$ if A then B
   It is not the case that if it will rain tomorrow then the Moon will fall onto the Earth. Therefore: It will rain tomorrow.
3. *Antec. Strength.* if A then $C \therefore$ if A and B then C
   If you add sugar to your coffee, it will taste good. Therefore: If you add sugar and vinegar to your coffee, it will taste good.
4. *Contraposition* if A then $B \therefore$ if not B then not A
   If our opponents are cheating, we will never find out. Therefore: If we will find out that our opponents are cheating, then they aren't cheating.

Strict conditionals?

Strict conditionals?

'If Russia invades Estonia, NATO will attack Russia.'

Perhaps this doesn't just say $r \rightarrow n$, but $\Box(r \rightarrow n)$.

$\Box(A \rightarrow B)$ is a **strict conditional**.

$\Box(A \rightarrow B)$ is true iff all accessible A-worlds are B-worlds.

Abbreviation: $A \prec\!\!3\ B$ iff all accessible A-worlds are B-worlds.

How should we understand the accessibility relation?

A popular assumption: wRv iff v is compatible with what is known at w.

Strict conditionals?

Modus Ponens is valid because epistemic accessibility is reflexive.

- Suppose $A \rightarrow\!\!\!\!3\ B$ and A.
- $\Box(A \rightarrow B)$ entails $A \rightarrow B$.
- $A \rightarrow B$ and A entail B.
- So B.

Strict conditionals?

		$A \rightarrow B$	$A \multimap B$
Modus Ponens	if *A* then *B*, *A* ∴ *B*	valid	valid
Conditional Proof	*A* entails *B* ∴ if *A* then *B*	valid	valid
Or-to-If	*A* ∨ *B* ∴ if not *A* then *B*	valid	invalid
Import-Export	if *A* then if *B* then *C* ∴∴ if *A* and *B* then *C*	valid	invalid
Contraposition	if *A* then *B* ∴ if not *B* then not *A*	valid	valid
Transitivity	if *A* then *B*, if *B* then *C* ∴ if *A* then *C*	valid	valid
SDA	if *A* or *B* then *C* ∴ if *A* then *C* and if *B* then *C*	valid	valid
Antec. Strength.	if *A* then *C* ∴ if *A* and *B* then *C*	valid	valid
False Antec.	not *A* ∴ if *A* then *B*	valid	invalid
True Cons.	*B* ∴ if *A* then *B*	valid	invalid

Problems:

- $A \strictif B \models \neg B \strictif \neg A$

  If our opponents are cheating, we will never find out. Therefore: If we will find out that our opponents are cheating, then they aren't cheating.

- $A \strictif B \models (A \wedge C) \strictif \neg B$

  If you add sugar to your coffee, it will taste good. Therefore: If you add sugar and vinegar to your coffee, it will taste good.

- $A \strictif B, B \strictif C \models A \strictif C$.

  If I quit my job, I won't be able to pay rent. If I win a million, I'll quit my job. Therefore: if I win a million, I won't be able to pay rent.

Strict conditionals?

Possible response:

The accessibility relation depends on conversational context.

- 'If you add sugar to your coffee, it will taste good.'
  – Here worlds where you add sugar and vinegar to your coffee are ignored/inaccessible.
- 'If you add sugar and vinegar to your coffee, it will taste good.'
  – Now these worlds are no longer ignored/inaccessible.

Indicative and subjunctive

Indicative and subjunctive

Indicative:

- If Hitler didn't commit suicide, he was hanged at Nuremberg.

Subjunctive/counterfactual:

- If Hitler hadn't committed suicide, he would have been hanged at Nuremberg.

Indicative conditionals *might* be material conditionals. Subjunctive conditionals definitely aren't.

Similarity semantics

- If Hitler hadn't committed suicide, he would have been hanged at Nuremberg.

Intuitively, to assess a subjunctive conditional, we

1. rewind the world to the time of the antecedent,
2. make minimal changes to render the antecedent true,
3. then let history run its course.

The conditional is true iff the consequent is true at all the resulting worlds.

Different antecedents call for different revisions to the actual world.

- If Hitler hadn't committed suicide …
- If Hitler had never been born …

We don't seem to consider *all* accessible *A*-worlds, but only the ones that most closely resemble the actual world (in relevant respects).

Similarity semantics

$A \mathbin{\square\!\!\rightarrow} B$ is true (at *w*) iff *B* is true at all the most similar *A*-worlds (to *w*).

Similarity semantics

A **similarity model** consists of

- a non-empty set W of worlds,
- for each world w in W a similarity order $\prec_w$, and
- a function V that assigns to each sentence letter a subset of W.

Similarity semantics for $\Box\!\to$

If M is a similarity model and w a world in M, then

$M, w \models A \Box\!\to B$ iff $M, v \models B$ for all v such that (i) $M, v \models A$ and (ii) there is no $u \prec_w v$ with $M, u \models A$.

Similarity semantics

		$A \rightarrow B$	$A \multimap B$	$A \,\square\!\!\rightarrow B$
Modus Ponens	if A then B, A ∴ B	valid	valid	valid
Conditional Proof	A entails B ∴ if A then B	valid	valid	valid
Or-to-If	A ∨ B ∴ if not A then B	valid	invalid	invalid
Import-Export	if A then if B then C ∴∴ if A and B then C	valid	invalid	invalid
Contraposition	if A then B ∴ if not B then not A	valid	valid	invalid
Transitivity	if A then B, if B then C ∴ if A then C	valid	valid	invalid
SDA	if A or B then C ∴ if A then C and if B then C	valid	valid	invalid
Antec. Strength.	if A then C ∴ if A and B then C	valid	valid	invalid
False Antec.	not A ∴ if A then B	valid	invalid	invalid
True Cons.	B ∴ if A then B	valid	invalid	invalid

If-clauses as restrictors

If-clauses as restrictors

(1) If the murderer escaped through the window, there must be traces on the ground.

(2) If the murderer escaped through the window, there might be traces on the ground.

(1) should not be translated as $p \rightarrow \Box q$ or $p \dashv3 \Box q$. But $\Box(p \rightarrow q)$ works.

(2) cannot be translated as $\Diamond(p \rightarrow q)$. Better: $p \rightarrow \Diamond q$. Even better: $\Diamond(p \wedge q)$.

If-clauses as restrictors

(1) If it rains, we always stay inside.
(2) If it rains, we sometimes stay inside.
(3) If it rains, we usually stay inside.

(1) can't be translated as $p \rightarrow \Box q$ or $p \dashv3 \Box q$. But $\Box(p \rightarrow q)$ works.

(2) can't be translated as $p \rightarrow \Diamond q$ or $\Diamond(p \rightarrow q)$. But $\Diamond(p \land q)$ works.

(3) can't be translated as $p \rightarrow Mq$ or $M(p \rightarrow q)$ or $M(p \land q)$ or ….

  (1) If it rains, we always stay inside.
  (2) If it rains, we sometimes stay inside.
  (3) If it rains, we usually stay inside.

(1) says that in all situations in which it rains, we stay inside.

(2) says that in some situations in which it rains, we stay inside.

(3) says that in most situations in which it rains, we stay inside.

(1) If the murderer escaped through the window, there must be traces on the ground.

(2) If the murderer escaped through the window, there might be traces on the ground.

(1) says that in all epistemically accessible worlds at which the murderer escaped through the window, there are traces on the ground.

(2) says that in some epistemically accessible worlds at which the murderer escaped through the window, there are traces on the ground.

(1) Jones should help his neighbours.

(2) If Jones won't help his neighbours, he shouldn't tell them that he is coming.

(1) says that in the best of the circumstantially accessible worlds, Jones helps his neighbours.

(2) says that in the best of the circumstantially accessible worlds at which Jones won't help his neighbours, Jones doesn't tell them that he is coming.

"The history of the conditional is the story of a syntactic mistake. There is no two-place *if...then* connective in the logical forms of natural languages. *If*-clauses are devices for restricting the domains of various operators. Whenever there is no explicit operator, we have to posit one."

— Angelika Kratzer, 1991

If-clauses as restrictors

(1) If Russia invades Estonia, NATO will attack Russia.

(1b) If Russia invades Estonia then it is certain that NATO will attack Russia.

(1b) says that at all epistemically accessible worlds at which Russia invades Estonia, NATO attacks Russia.

This is equivalent to $\Box(r \rightarrow n)$, with an epistemic accessibility relation.

If-clauses as restrictors

(2) If Hitler hadn't committed suicide, he would have been hanged.

Perhaps 'would' is a modal operator, meaning something like 'it is settled that'.

- She wrote a book. It would later become a bestseller.

Suppose 'would q' is true iff the laws of nature together with the current facts entail q.

So 'would q' is true at w iff q is true at all the closest worlds to w.

'If p would q' is true at w iff q is true at all the closest *p*-worlds to w.

This is equivalent to $p \mathbin{\Box\!\!\rightarrow} q$.